Key references to the linear theory

References

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3. Minimal and large positive solutions.

Assume $-\Delta + V$ satisfies λ-property and $\Omega = \mathbb{R}^N \setminus B_1 = B_1^c$ — exterior domain, $R \geq 1$

Lemma 1. $\exists$ unique "finite energy" solution u_1 to

$-\Delta u + Vu = 0$ in B_R^c, $u = 1$ on $|x| = R$. (*)

◀ Take a smooth φ: $\varphi = 1$ on $|x| = R$, $\varphi = 0$ on $|x| > 2R$

Set $-\Delta\varphi + V\varphi = F \in (\mathcal{D}_V^1(B_R^c))^*$.

Solve $-\Delta v + Vv = -F$ in $\mathcal{D}_V^1(B_R^c)$.

Then $u = v + \varphi$ solves (*). ▶

u_1 is the minimal positive solution at infinity for $-\Delta+V$ in B_1^c:

$\forall$ positive supersolution w in B_1^c $\exists c>0$:

$w \geq c u_1$ in B_R^c.

$-\Delta w + Vw \geq 0$ in B_1^c

◀ [Agmon, Th. 4.2] ▶

Lemma 2. $\exists$ unique (up to a scalar) positive sol. u_0 to

$$-\Delta u + Vu = 0 \quad \text{in } B_R^c, \qquad u = 0 \quad \text{on } |x| = R$$

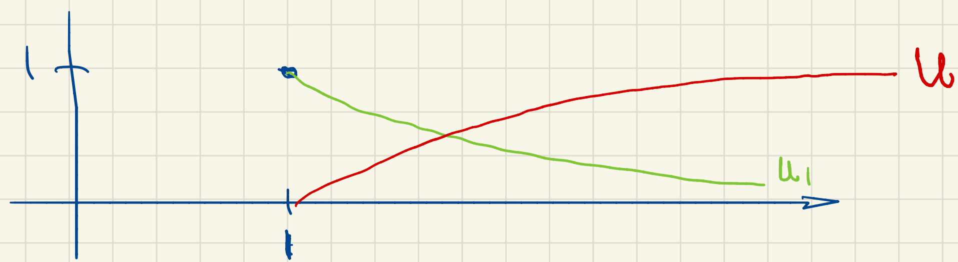
$\triangleleft$ [Agmon, Th. 3.1] $\triangleright$

$u_0 \notin \mathcal{D}^1_V(B_R^c)$!

$E_V(u_0) = 0$

Example: $-\Delta$ in B_1^c, $N \geq 3$

$$u_1 = |x|^{2-N}, \qquad u_0 = 1 - |x|^{2-N}$$



Def. We say V is a small subsolution at infinity for $-\Delta + V$ if $\exists$ a supersolution $u_* > 0$ such that $\lim_{|x|\to\infty} \frac{v(x)}{u_*(x)} = 0.$

We say V is a large subsolution at infinity for $-\Delta + V$ if V is not a small subsol., i.e. $\forall$ supersolution $u > 0$, $\limsup_{|x|\to\infty} \frac{v(x)}{u_*(x)} > 0.$

Lemma 3. If V is a small subsolution at ∞, then for any positive supersolution $u > 0$ $\exists\, c > 0$: $u \geq cV$ in B_R^c

◀ We may assume $\sup\limits_{|x|=R} V = 1$. Set $c := \inf\limits_{|x|=R} u > 0$

Consider $V_\varepsilon := cV - \varepsilon u_*$. Then

1) $V_\varepsilon < u$ on $|x| = R$

2) $V_\varepsilon = 0$ on $|x| > \rho_\varepsilon \gg R$

$\Rightarrow$ apply comparison on A_{R,ρ_ε} ▷

$\Longrightarrow u \geq V_\varepsilon \xrightarrow{\varepsilon \to 0}$ $\Longrightarrow u \geq$

Lemma 4. Assume that

$$-\Delta V + V v \leq 0 \text{ in } B_R^c, \quad V = 0 \text{ on } |x| = R.$$

Then V is a large subsolution.

◀ Assume V is not large

$\exists\, u_* > 0, \quad -\Delta u_* + V u_* \geq 0$ in B_R^c,

$$\lim_{|x|\to\infty} \frac{V}{u_*} = 0.$$

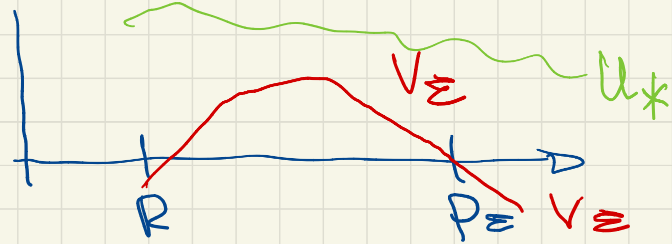
As before, consider $V_\varepsilon = V - \varepsilon u_*$.

Then $V_\varepsilon \leq 0$ on $|x| = R$ and $|x| = \rho_\varepsilon \geq R$,

and $V_\varepsilon^+ \neq 0$

Fix ε, and take

$$V_n = n V_\varepsilon, \quad \forall n \in \mathbb{N}$$



By Comparison, $V_n \leq u_* \quad \forall n \in \mathbb{N}$

But $\forall$ compact $K \subset \operatorname{supp}(V_\varepsilon)$,

$$\sup_n V_n(x) = +\infty \quad \forall x \in K$$

$\Rightarrow u_* = +\infty$ on K, a contradiction

Phragmen-Lindelöf type principle:

Let $u>0$ be a supersol to $-\Delta+V$ in B_1^c.

Then: 1) for any small subsol. V at ∞
$\exists R, c>0$: $\quad u \geq cV$ in $|x|>R>1$

2) for any large subsolution W at ∞

$$\liminf_{|x|\to\infty} \frac{V}{W} < +\infty$$

[Protter-Weinberger]

1) = Lemma 3
2) = Def of large subsol.

Examples:

1) $-\Delta$ on $\mathbb{R}^N \setminus B_1$, $N \geq 3$

$u_1 = |x|^{-(N-2)}$ is a small (sub)solution

◀ $u_* = |x|^{-\frac{N-2}{2}}$ $\qquad \lim_{|x| \to \infty} \frac{u_1}{u_*} = 0$ ▶

$-\Delta u_* \geq 0$

$u_0 = 1 - |x|^{-(N-2)}$ is a large (sub)sol.

◀ Lemma 4. ▶

2) $-\Delta$ on $\mathbb{R}^2 \setminus B_1$

a) $u_1 = 1$ is a small (~~sub~~)sol.

$$u_* = (\log|x|)^{\frac{1}{2}} > \qquad \frac{u_1}{u_*} \xrightarrow[|x|\to\infty]{} 0$$

$-\Delta u_* \geq 0$

b) $u_0 = \log|x|$ is a large ~~sub~~sol in B_1^c

$-\Delta u_0 = 0$

$$-\Delta u \geq 0 \text{ in } B_1^c \implies$$

$$0 < \liminf_{|x|\to\infty} u(x), \quad \liminf_{|x|\to\infty} \frac{u(x)}{\log|x|} < +\infty$$

Remark. $\mathcal{D}_0^1(\mathbb{R}^2 \setminus B_1)$ contains functions that do not decay to 0 at ∞!

$$\begin{cases} -\Delta u_f = f \geq 0 \text{ in } |x|>1, \\ u_f \text{ "="} \ 0 \text{ on } |x| = 0 \end{cases} \qquad \forall f \in \left(\mathcal{D}_0^1(B_1^c)\right)^*$$

$$u_f \in \mathcal{D}_0^1(B_1^c)$$

$$\liminf_{|x|\to\infty} u_f > 0 \qquad u_f \notin L^p(B_1^c) \quad \forall 1 \leq p < \infty$$

Example: $-\Delta + V$ in $\mathbb{R}^N \setminus B_1$, $N \geq 3$, $V \in L^\infty(\mathbb{R}^N)$

$$\lim_{|x|\to\infty} \frac{|V(x)|}{|x|^{2+\varepsilon}} = 0$$ and $-\Delta + V$ satisfies λ-property

Then $-\Delta u + Vu \geq 0$ in B_1^c, $u > 0 \Longrightarrow$

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|^{N-2}} > 0, \quad \liminf_{|x|\to\infty} u(x) < +\infty$$

– the same as for $-\Delta$!

Take $V_1(x) = |x|^{-(N-2)} + a|x|^{-(N-2)-\delta}$, $a, \delta > 0$ small

— small sub-solution

$$V_0(x) = 1 - a|x|^{-(N-2)+\delta}, \quad -(N-2)+\delta < 0$$

— large subsolution (adjust at $|x| = R$)

$-\Delta V_1 + V(x) V_1 \leq 0$, for some small $a > 0$

$$V_1(x) \simeq |x|^{-(N-2)}\left(1 + o(1)\right) \text{ as } |x| \to \infty$$

$$V_0(x) \simeq 1 + o(1)$$

$-\Delta V_0 + V(x) V_0 \leq 0$ in $|x| > R$, $V_0 = 0$ on $|x| = R$

3) $-\Delta - \frac{c}{|x|^2}$ in $\mathbb{R}^N \setminus B_1$, $N \geq 3$ Exercise

$-\infty < c < c_H = \left(\frac{N-2}{2}\right)^2$

$\Rightarrow$ $|x|^{\alpha_-}$ — small solution,

$|x|^{\alpha_+} - |x|^{\alpha_-}$ — large solution,

$\alpha_- < \alpha_+$ — roots of $-\alpha(\alpha+N-2) = c$